INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 42, Northern Autumn 2020 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. There are n positive integers. For each pair of those integers Boris wrote their arithmetic mean on a blackboard and their geometric mean on a whiteboard. It turns out that for each pair at least one of those means is an integer. Prove that on at least one of the boards all the numbers are integers. (4 points)

2. Baron Munchausen presents a new theorem: if a polynomial $x^n - ax^{n-1} + bx^{n-2} + \ldots$ has n positive integer roots then there exist a lines in the plane such that they have exactly b intersection points. Is this theorem true? (5 points)

3. Two circles α and β with centers A and B respectively intersect at points C and D. The line segment AB intersects α and β at points K and L respectively. The ray DK intersects the circle β for the second time at the point N, and the ray DL intersects the circle α for the second time at the point M. Prove that the intersection point of the diagonals of the quadrilateral $KLMN$ coincides with the incenter of the triangle ABC. (6 points)

4. There are two round tables with n dwarves sitting at each table. Each dwarf has only two friends: his neighbours to the left and to the right. A kind wizard wants to seat the dwarves at one round table so that every two neighbours are friends. His magic allows him to make any $2n$ pairs of dwarves into pairs of friends (the dwarves in a pair may be from the same or from different tables). However, he knows that an evil sorcerer will break n of those new friendships. For which values of n can the kind wizard achieve his goal no matter what the evil sorcerer does?

   (7 points)

5. Does there exist a rectangle which can be cut into a hundred smaller rectangles such that all of them are similar to the original one but no two of them are congruent?

   (7 points)

6. Alice and Bob play the following game. They take turns with Alice starting the game. On each turn a player writes a fraction of the form $1/n$, where n is a positive integer, on the blackboard. On her turn, Alice always writes one fraction, but Bob writes one fraction on his first turn, two fractions on his second turn, three fractions on his third turn and so on. Bob wants to make the sum of all the fractions on the board to be an integer number after some turn. Can Alice prevent this?

(10 points)

7. There is a grid consisting of $1000 \times n$ square cells where n is odd and $n > 2020$. A white bug sits in one of the corner squares of this grid. In the two nearest corner squares there are two black chess bishops. The bug can either move one cell left, right, up or down, or it moves as a chess knight. The bug wants to reach the opposite corner cell by never visiting cells which are either occupied or attacked by bishops, and visiting every other square exactly once. Show that the number of ways for the bug to attain its goal does not depend on n.

(12 points)